Computability

Computable Functions, Logic, and the Foundations of Mathematics

3rd edition

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Preface

Why was the theory of computable functions developed before there were any computers?

The formal theory of computable functions and their relation to logic arose as a response to the ferment in the foundations of mathematics at the beginning of this century. The paradoxes of self-reference and the question of how or even whether we are justified in using infinite sets stood at the center of that development, and those paradoxes are no less interesting, nor settled, now. Along with readings from the originators of the subject, the paradoxes and doubts about the infinite serve to motivate the study of the technical mathematics in this book and place the mathematics in its history.

Some mathematicians may prefer a straight mathematical development; for that Part II, *Computable Functions*, and Part III, *Logic and Arithmetic*, will suffice. In Part II we describe the notion of computability, present the Turing machine model, and then develop the theory of partial recursive functions as far as the Normal Form Theorem. In Part III we begin with propositional logic and give an overview of predicate logic and Gödel's theorems, which can serve as a summary for a short course. A full development of the syntactic part of first-order logic and Gödel's theorems then follows. Part I, *The Fundamentals*, can be referred to for notation and basic proof techniques.

Philosophy, however, has been the motive for much of logic and computability. In Part I we give the philosophical background for discussions about the foundations of mathematics while presenting the notions of whole number, function, proof, and real number. Hilbert's paper "On the infinite" sets the stage for the analysis of computability in Part II. In Part IV we consider the significance of the technical work with discussions of Church's Thesis, constructivity in mathematics, and mathematics as modeling.

Many exercises are included, beginning gently in Part I and progressing to a graduate level in the final chapters. The most difficult ones, marked with a dagger †, may be skipped, although all are intended to be read. Solutions to the exercises can be found in the Instructor's Manual (available from the Advanced Reasoning Forum, www.AdvancedReasoningForum.org), which also contains suggestions for course outlines. Sections marked "Optional" are not essential for the technical development of chapters which follow, although they often provide important motivation.

For the second edition: Beyond the addition of a timeline on computability and undecidability written by Epstein, we have confined our changes almost entirely to technical corrections, adding only two new quotes from Gödel (p. 173 and p. 215). One noteworthy change is the replacement of Fermat's Last Theorem by Goldbach's Conjecture as an example of an unsolved arithmetic problem used in several examples; the former has been shown to be true by Andrew Wiles, "Modular elliptic curves and Fermat's Last Theorem", *Annals of Mathematics*, second series, vol. 141, 1995, pp. 443–551. Where other authors have used Fermat's Last Theorem as an example (Arend Heyting on p. 234, Nicholas Goodman on p. 259), a similar substitution of Goldbach's Conjecture would make the same point.

For the third edition: We have added a chapter that gives a very different view of mathematics than in the other articles in the text, viewing mathematics as modeling and not (necessarily) in need of foundations. It is the view that underlies our presentation of the mathematics in this book.

We have made only minor corrections to the body of the text, retaining the same pagination. A few small corrections have been made to the timeline.

The story we tell leaves no room to include a presentation of the semantics of classical predicate logic. That material is now available in a companion to this volume, Epstein's *Classical Mathematical Logic*, Princeton University Press, 2006.

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Each author wishes to indicate that any mistakes still left in this text are not due to those above who have so generously helped us, but are due entirely to the other author.

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(Full references to the papers and books below may be found in the Bibliography.)

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"On the infinite," by David Hilbert, was first delivered on June 4, 1925, before a congress of the Westphalian Mathematical Society in Münster, in honor of Karl Weierstrass. Translated by Erna Putnam and Gerald J. Massey from *Mathematischen Annalen* (Berlin) no. 95 (1926) pp. 161–190. Permission for the translation and inclusion of the article in this volume was kindly granted by the publishers, Springer–Verlag.

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A previous version of Chapter 27 appeared at CLE e-Prints, Vol. 8(3), 2001 (Section Logic), /www.cle.unicamp.br/e-prints/vol_8,n_3,2008.html.