

Abstract

Although most of the familiar logical systems are known to have an algebraic counterpart, no general and precise notion of an algebraizable logic exists upon which a systematic investigation of the process of algebraization can be based. In the memoir such a notion is proposed and the investigation begun.

A deductive system \mathcal{S} over a language \mathcal{L} is *algebraizable* if there exists a quasivariety \mathbf{K} of \mathcal{L} -algebras such that the \mathcal{S} -consequence relation $\vdash_{\mathcal{S}}$ and the equational consequence relation $\models_{\mathbf{K}}$ over \mathbf{K} are interpretable in one another in a certain strong sense; \mathbf{K} is called an *equivalent algebraic semantics for \mathcal{S}* . If \mathcal{S} is algebraizable, then it has precisely one equivalent algebraic semantics. All the logical systems that were known to have an algebraic representation prove to be algebraizable in this precise sense, and in each case the algebraic counterpart turns out to be the equivalent algebraic semantics (up to definitional equivalence).

The main result of the paper is an intrinsic characterization of algebraizability in terms of the *Leibniz operator* Ω , which associates with each theory T of a given deductive system \mathcal{S} a congruence relation ΩT on the formula algebra. ΩT identifies all formulas that cannot be distinguished from one another, on the basis of T , by any property expressible in the language of \mathcal{S} . The characterization theorem states that a deductive system \mathcal{S} is algebraizable if and only if Ω is one-to-one and order-preserving on the lattice of \mathcal{S} -theories, and in addition preserves directed unions. Several other characterizations are given.

The results and concepts are illustrated by a large number of examples from modal and intuitionistic logic, relevance logic, and classical predicate logic.

Received by the editors April 19, 1987.

Key-words: deductive system, consequence relation, formula algebra, equational consequence, quasivariety, lattice of theories, universal Horn theory, modal logic, intuitionistic logic, relevance logic, predicate logic.

Contents

Introduction	1
1 Deductive Systems and Matrix Semantics	5
1.1 The Lattice of Theories	6
1.2 Matrix Semantics	8
1.3 Deductive Systems as Elementary Theories	9
1.4 The Elementary Leibniz Equivalence Relation	10
1.4.1 Protoalgebraic Logics	12
2 Equational Consequence and Algebraic Semantics	13
2.1 Algebraic Semantics	14
2.2 Equivalent Algebraic Semantics	19
2.2.1 Uniqueness	22
2.2.2 Axiomatization	24
3 The Lattice of Theories	27
4 Two Intrinsic Characterizations	34
4.1 The Leibniz Operator	34
4.2 A Second Intrinsic Characterization	39
5 Matrix Semantics and Algebraizability	42
5.1 Matrix Semantics and Algebraic Semantics	42
5.2 Applications and Examples	46
5.2.1 Modal Logics	46
5.2.2 Entailment and Relevance Logics	48
5.2.3 Pure Implicational Logics	49
5.2.4 Two Logics with the Same Algebraization	54
5.2.5 Intuitionistic Propositional Logic without Implication	56
5.2.6 Equivalential Logic	56

A Elementary Definitional Equivalence	60
B An Example	63
C Predicate Logic	67
Bibliography	73
Index	77