Abstract

Although most of the familiar logical systems are known to have an algebraic counterpart, no general and precise notion of an algebraizable logic exists upon which a systematic investigation of the process of algebraization can be based. In the memoir such a notion is proposed and the investigation begun.

A deductive system $\mathcal{S}$ over a language $\mathcal{L}$ is *algebraizable* if there exists a quasivariety K of $\mathcal{L}$-algebras such that the $\mathcal{S}$-consequence relation $\vdash_{\mathcal{S}}$ and the equational consequence relation $\models_{\mathsf{K}}$ over K are interpretable in one another in a certain strong sense; K is called an *equivalent algebraic semantics for* $\mathcal{S}$. If $\mathcal{S}$ is algebraizable, then it has precisely one equivalent algebraic semantics. All the logical systems that were known to have an algebraic representation prove to be algebraizable in this precise sense, and in each case the algebraic counterpart turns out to be the equivalent algebraic semantics (up to definitional equivalence).

The main result of the paper is an intrinsic characterization of algebraizability in terms of the *Leibniz operator* Ω, which associates with each theory T of a given deductive system $\mathcal{S}$ a congruence relation ΩT on the formula algebra. ΩT identifies all formulas that cannot be distinguished from one another, on the basis of T, by any property expressible in the language of $\mathcal{S}$. The characterization theorem states that a deductive system $\mathcal{S}$ is algebraizable if and only if Ω is one-to-one and order-preserving on the lattice of $\mathcal{S}$-theories, and in addition preserves directed unions. Several other characterizations are given.

The results and concepts are illustrated by a large number of examples from modal and intuitionistic logic, relevance logic, and classical predicate logic.

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